Selected results on selection principles

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Abstract

We review some selected recent results concerning selection principles in topology and their relations with several topological constructions.

1 Introduction

The beginning of investigation of covering properties of topological spaces defined in terms of diagonalization and nowadays known as *classical selection principles* is going back to the papers [52], [35], [36], [63]. In this paper we shall briefly discuss these classical selection principles and their relations with other fields of mathematics, and after that we shall be concentrated on recent innovations in the field, preferably on results which are not included into two nice survey papers by M. Scheepers [73], [74]. In particular, in [74] some information regarding "modern", non-classical selection principles can be found. No proofs are included in the paper.

Two classical selection principles are defined in the following way.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are families of subsets of an infinite set X. Then:

$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

> For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subset A_n$, and $\bigcup_{n \in \mathbb{N}} B_n$ is an element of $\mathcal{B}$.

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

> For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

In [52] Menger introduced a property for metric spaces (called now the *Menger basis property*) and in [35] Hurewicz has proved that that property is equivalent to the property $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$, where $\mathcal{O}$ denotes the family of open covers of the space, and called now the *Menger property*.

In the same paper (see also [36]) Hurewicz introduced another property, nowadays called the *Hurewicz property*, defined as follows. A space X has the Hurewicz property if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each element of the space belongs to all but finitely many of the sets $\cup \mathcal{V}_n$. It was shown in [48] that the Hurewicz property can be expressed in terms of a selection principle of the form $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$.

A selection principle of the $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ type was introduced in the 1938 Rothberger's paper [63], in connection with his study of strong measure zero sets in metric spaces that were first defined by Borel in [16]. The *Rothberger property* is the property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$.

Other two properties of this sort were introduced by Gerlits and Nagy in [30] under the names γ-*sets* and *property* ($*$) (the later is of the $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ kind as it was shown in [48]).

The collections $\mathcal{A}$ and $\mathcal{B}$ that we consider here will be mainly families of open covers of some topological space. We give now the definitions of open covers which are important for this exposition.

An open cover $\mathcal{U}$ of a space X is:

- an ω-*cover* if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$ [30].

- a k-*cover* if X does not belong to $\mathcal{U}$ and every compact subset of X is contained in a member of $\mathcal{U}$ [51].

- a γ-*cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$ [30].

- a γ_k-*cover* if each compact subset of X is contained in all but finitely many elements of $\mathcal{U}$ and X is not a member of the cover [43].

- *large* if each $x \in X$ belongs to infinitely many elements of $\mathcal{U}$ [69].

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- *groupable* if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that each $x \in X$ belongs to $\cup\mathcal{U}_n$ for all but finitely many n [48].

- ω-*groupable* if it is an ω-cover and is a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that each finite subset of X is contained in some element U in $\mathcal{U}_n$ for all but finitely many n [48].

- *weakly groupable* if it is a countable union of finite, pairwise disjoint sets $\mathcal{U}_n$, $n \in \mathbb{N}$, such that for each finite set $F \subset X$ we have $F \subset \cup\mathcal{U}_n$ for some n [7].

- a τ-*cover* if it is large and for any two distinct points x and y in X either the set $\{U \in \mathcal{U} : x \in U \text{ and } y \notin U\}$ is finite, or the set $\{U \in \mathcal{U} : y \in U \text{ and } x \notin U\}$ is finite [81].

- a τ^*-*cover* if it is large and for each x there is an infinite set $A_x \subset \{U \in \mathcal{U} : x \in U\}$ such that whenever x and y are distinct, then either $A_x \setminus A_y$ is finite, or $A_y \setminus A_x$ is finite [76].

For a topological space X we denote:

- Ω – the family of ω-covers of X;

- $\mathcal{K}$ – the family of k-covers of X;

- Γ – the family of γ-covers of X;

- Γ_k – the family of γ_k-covers of X;

- Λ – the family of large covers of X;

- $\mathcal{O}^{gp}$ – the family of groupable covers of X;

- Λ^{gp} – the family of groupable large covers of X;

- Ω^{gp} – the family of ω-groupable covers of X;

- $\mathcal{O}^{wgp}$ – the family of weakly groupable covers of X;

- T – the set of τ-covers of X;

- T^* – the set of τ^*-covers of X.

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All covers that we consider are *infinite and countable* (spaces whose each ω-cover contains a countable subset which is an ω-cover are called ω-*Lindelof* or ϵ-*spaces* and spaces whose each k-cover contains a countable subset that is a k-cover are called k-*Lindelöf*).

So we have

$$\Gamma \subset \mathsf{T} \subset \mathsf{T}^* \subset \Omega \subset \Lambda \subset \mathcal{O},$$
$$\Gamma_k \subset \Gamma \subset \Omega^{gp} \subset \Omega \subset \Lambda^{wgp} \subset \Lambda \subset \mathcal{O},$$
$$\Gamma_k \subset \mathcal{K} \subset \Omega.$$

In this notation, according to the definitions and results mentioned above, we have:

- The Menger property: $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$;

- The Rothberger property: $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$;

- The Hurewicz property: $\mathsf{S}_{fin}(\Omega, \Lambda^{gp})$;

- The γ-set property: $\mathsf{S}_1(\Omega, \Gamma)$;

- The Gerlits-Nagy property $(*)$: $\mathsf{S}_1(\Omega, \Lambda^{gp})$.

It is also known:

- $X \in \mathsf{S}_{fin}(\Omega, \Omega)$ iff $(\forall n \in \mathbb{N})\, X^n \in \mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ [37];

- $X \in \mathsf{S}_1(\Omega, \Omega)$ iff $(\forall n \in \mathbb{N})\, X^n \in \mathsf{S}_1(\mathcal{O}, \mathcal{O})$ [65];

- $X \in \mathsf{S}_{fin}(\Omega, \Omega^{gp})$ iff $(\forall n \in \mathbb{N})\, X^n \in \mathsf{S}_{fin}(\Omega, \Lambda^{gp})$ [48];

- $X \in \mathsf{S}_1(\Omega, \Gamma)$ iff $(\forall n \in \mathbb{N})\, X^n \in \mathsf{S}_{fin}(\Omega, \Gamma)$ [30];

- $X \in \mathsf{S}_1(\Omega, \Omega^{gp})$ iff $(\forall n \in \mathbb{N})\, X^n \in \mathsf{S}_1(\Omega, \Lambda^{gp})$ [48].

For a space X and a point $x \in X$ the following notation will be used:

- Ω_x – the set $\{A \subset X \setminus \{x\} : x \in \overline{A}\}$;

- Σ_x – the set of all nontrivial sequences in X that converge to x.

A countable element $A \in \Omega_x$ is said to be *groupable* [48] if it can be expressed as a union of infinitely many finite, pairwise disjoint sets B_n, $n \in \mathbb{N}$, such that each neighborhood U of x intersects all but finitely many sets B_n. We put:

- Ω_x^{gp} – the set of groupable elements of Ω_x.

Games

Already Hurewicz observed that there is a natural connection between the Menger property and an infinitely long game for two players. In fact, in [35] Hurewicz implicitly proved that the principle $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ is equivalent to a game theoretical statement (ONE does not have a winning strategy in the game $\mathsf{G}_{fin}(\mathcal{O}, \mathcal{O})$; see the definition below and for the proof see [69]).

Let us define games which are naturally associated to the selection principles $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ and $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ introduced above.

Again, $\mathcal{A}$ and $\mathcal{B}$ will be sets whose elements are families of subsets of an infinite set X.

$\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$ denotes an infinitely long game for two players, ONE and TWO, which play a round for each positive integer. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing a finite set $B_n \subset A_n$. The play $(A_1, B_1, \cdots, A_n, B_n, \cdots)$ is won by TWO if $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$; otherwise, ONE wins.

$\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ denotes a similar game, but in the n-th round ONE chooses a set $A_n \in \mathcal{A}$, while TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \cdots; A_n, b_n; \cdots)$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

It is evident that if ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ (resp. $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$) then the selection hypothesis $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ (resp. $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$) is true. The converse implication need not be always true.

We shall see that a number of selection principles we mentioned can be characterized by the corresponding game (see Table 1).

Ramsey theory

Ramsey Theory is a part of combinatorial mathematics which deals with partition symbols. In 1930, F.P. Ramsey proved the first important partition theorems [61]. Nowadays there are many "partition symbols" that have been extensively studied. We shall consider here two partition relations (the ordinary partition relation and the Baumgartner-Taylor partition relation) which have nice relations with classical selection principles and infinite game theory in topology. M. Scheepers was the first who realized these connections (see [69]). In [48] very general results of this sort were given. They show how to derive Ramsey-theoretical results from game-theoretic statements, and how selection hypotheses can be derived from Ramseyan partition relations. For a detail exposition on applications of Ramsey theory to topological properties see [44].

We shall also list several results which demonstrate how some closure properties of function spaces can be also described Ramsey theoretically and game theoretically (see Table 2).

Let us mention that no Ramseyan results are known for non-classical selection principles that appeared in the literature in recent years.

We are going now to define the two partition relations we shall do with.

For a set X the symbol $[X]^n$ denotes the set of n-element subsets of X, while $\mathcal{A}$ and $\mathcal{B}$ are as in the definitions of selection hypotheses and games. Let n and m be positive integers. Then:

The *ordinary partition symbol* (or *ordinary partition relation*)

$$\mathcal{A} \to (\mathcal{B})^n_m$$

denotes the statement:

> For each $A \in \mathcal{A}$ and for each function $f : [A]^n \to \{1, \cdots, m\}$ there are a set $B \in \mathcal{B}$ with $B \subset A$ and some $i \in \{1, \cdots, m\}$ such that for each $Y \in [B]^n$, $f(Y) = i$.

The *Baumgartner-Taylor partition symbol* [13]

$$\mathcal{A} \to \lceil \mathcal{B} \rceil^2_m$$

denotes the following statement:

> For each A in $\mathcal{A}$ and for each function $f : [A]^2 \to \{1, \cdots, m\}$ there are a set $B \in \mathcal{B}$ with $B \subset A$, an $i \in \{1, \cdots, m\}$ and a partition $B = \bigcup_{n \in \mathbb{N}} B_n$ of B into pairwise disjoint finite sets such that for each $\{x, y\} \in [B]^2$ for which x and y are not from the same B_n, we have $f(\{x, y\}) = i$.

Several selection principles of the form $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ (resp. $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$) can be characterized by the ordinary (resp. the Baumgartner-Taylor) partition relation (see Tables 1 and 2).

Our topological notation and terminology are standard and follow those from [23] with one exception: in Section 3 and Section 4 Lindelöf spaces are not supposed to be regular. All spaces are assumed to be *infinite and Hausdorff*. (Notice that some of results which will be mentioned here hold for wider classes of spaces than it is indicated in our statements.) For a Tychonoff space X $\mathsf{C}_p(X)$ (resp. $\mathsf{C}_k(X)$) denotes the space of all continuous real-valued functions on X with the topology of pointwise convergence (resp.

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the compact-open topology). $\underline{0}$ denotes the constantly zero function from $\mathsf{C}_p(X)$ and $\mathsf{C}_k(X)$. Some notions we are doing with will be defined when they become necessary.

The paper is organized in the following way. In Section 2 we give results showing relationships between selection principles, game theory and partition relations, as well as showing duality between covering properties of a space X and function spaces $\mathsf{C}_p(X)$ and $\mathsf{C}_k(X)$ over X. Section 3 is devoted to duality between covering properties of a space X (expressed in terms of selection principles) and properties of hyperspaces over X that appeared recently in the literature. In Section 4 we discuss star selection principles – an innovation in selection principles theory. In particular, we discuss selection principles in uniform spaces and topological groups. Finally, Section 5 contains some results concerning another innovation in the field – relative selection principles. Several open problems are included in each section. For a detail exposition about open problems we refer the interested reader to [82].

2   Selection principles, games, partition relations

Relationships of classical selection principles with the corresponding games and partition relations are given in the following table. "Game" means "ONE has no winning strategy", n and m are positive integers and "Source" gives papers in which results were originally shown - the first for games and the second for partition relations.

In Table 2 we give some results concerning relations between covering properties of a Tychonoff space X and closure properties of the function space $\mathsf{C}_p(X)$ over X.

Let us recall that a space X has *countable tightness* (resp. the *Fréchet-Urysohn property* FU) if for each $x \in X$ and each $A \in \Omega_x$ there is a countable set $B \subset A$ with $B \in \Omega_x$ (resp. a sequence $(x_n : n \in \mathbb{N})$ in A converging to x). X is SFU (*strictly FU*) if it satisfies $\mathsf{S}_1(\Omega_x, \Sigma_x)$ for each $x \in X$. X has *countable fan tightness* [2] (resp. *countable strong fan tightness* [65]) if it satisfies $\mathsf{S}_{fin}(\Omega_x, \Omega_x)$ (resp. $\mathsf{S}_1(\Omega_x, \Omega_x)$) for each $x \in X$. X has the *Reznichenko property* (E. Reznichenko, 1996) [47], [48] if for every $x \in X$ each $A \in \Omega_x$ contains a countable set $B \subset A$ with $B \in \Omega_x^{gp}$.

	Selection Property	Game	Partition relation	Source
1	$S_1(\mathcal{O},\mathcal{O})$	$G_1(\mathcal{O},\mathcal{O})$	$\Omega \to (\Lambda)_m^2$	[58], [69]
2	$S_1(\Omega,\Omega)$	$G_1(\Omega,\Omega)$	$\Omega \to (\Omega)_m^n$	[70], [69]
3	$S_1(\mathfrak{B},\mathfrak{B})$	$G_1(\mathfrak{B},\mathfrak{B})$	$\mathfrak{B}_\Omega \to (\mathfrak{B})_m^2$	[75]
4	$S_1(\mathcal{K},\mathcal{K})$	?	$\mathcal{K} \to (\mathcal{K})_2^2$	[20]
5	$S_1(\Omega,\Gamma)$	$G_1(\Omega,\Gamma)$	$\Omega \to (\Gamma)_m^2$	[30], [69]
6	$S_1(\mathcal{K},\Gamma)$	$G_1(\mathcal{K},\Gamma)$	$\mathcal{K} \to (\Gamma)_m^n$	[20], [17]
7	$S_1(\mathcal{K},\Gamma_k)$	$G_1(\mathcal{K},\Gamma_k)$	$\mathcal{K} \to (\Gamma_k)_m^n$	[43], [17]
8	$S_1(\Omega,\Lambda^{gp})$	$G_1(\Omega,\Lambda^{gp})$	$\Omega \to (\Lambda^{gp})_m^n$	[48]
9	$S_1(\Omega,\Omega^{gp})$	$G_1(\Omega,\Omega^{gp})$	$\Omega \to (\Omega^{gp})_m^n$	[48]
10	$S_1(\Omega,\mathcal{O}^{wgp})$	$G_1(\Omega,\mathcal{O}^{wgp})$	$\Omega \to (\mathcal{O}^{wgp})_m^2$	[7]
11	$S_{fin}(\mathcal{O},\mathcal{O})$	$G_{fin}(\mathcal{O},\mathcal{O})$	$\Omega \to \lceil\mathcal{O}\rceil_m^2$	[35], [69]
12	$S_{fin}(\Omega,\Omega)$	$G_{fin}(\Omega,\Omega)$	$\Omega \to \lceil\Omega\rceil_m^2$	[70], [69]+ [37]
13	$S_{fin}(\mathcal{K},\mathcal{K})$	?	$\mathcal{K} \to \lceil\mathcal{K}\rceil_2^2$	[20]
14	$S_{fin}(\Omega,\Lambda^{gp})$	$G_{fin}(\Omega,\Lambda^{gp})$	$\Omega \to \lceil\Lambda^{gp}\rceil_m^2$	[48]
15	$S_{fin}(\Omega,\Omega^{gp})$	$G_{fin}(\Omega,\Omega^{gp})$	$\Omega \to \lceil\Omega^{gp}\rceil_m^2$	[48]
16	$S_{fin}(\Omega,\Lambda^{wgp})$	$G_{fin}(\Omega,\Lambda^{wgp})$	$\Omega \to \lceil\Lambda^{wgp}\rceil_m^2$	[7]
17	$S_{fin}(\Omega,(T^*)^{gp})$	$G_{fin}(\Omega,(T^*)^{gp})$	$\Omega \to \lceil(T^*)^{gp}\rceil_m^2$	[76]
18	$S_1(\Omega,(T^*)^{gp})$	$G_1(\Omega,(T^*)^{gp})$	$\Omega \to ((T^*)^{gp})_m^2$	[76]

Table 1

	$C_p(X)$	X	$X^n\,(\forall n \in \mathbb{N})$	Source
1	Countable tightness	ω-Lindelöf	Lindelöf	[3]
2	FU	$S_1(\Omega,\Gamma)$	$S_1(\Omega,\Gamma)$	[30]
3	SFU	$S_1(\Omega,\Gamma)$	$S_1(\Omega,\Gamma)$	[30]
4	$S_{fin}(\Omega_{\underline{0}},\Omega_{\underline{0}})$	$S_{fin}(\Omega,\Omega)$	$S_{fin}(\mathcal{O},\mathcal{O})$	[2]
5	$G_{fin}(\Omega_{\underline{0}},\Omega_{\underline{0}})$	$S_{fin}(\Omega,\Omega)$	$S_{fin}(\mathcal{O},\mathcal{O})$	[70]
6	$\Omega_{\underline{0}} \to \lceil\Omega_{\underline{0}}\rceil_2^2$	$S_{fin}(\Omega,\Omega)$	$S_{fin}(\mathcal{O},\mathcal{O})$	[70]
7	$S_1(\Omega_{\underline{0}},\Omega_{\underline{0}})$	$S_1(\Omega,\Omega)$	$S_1(\mathcal{O},\mathcal{O})$	[65]
8	$G_1(\Omega_{\underline{0}},\Omega_{\underline{0}})$	$S_1(\Omega,\Omega)$	$S_1(\mathcal{O},\mathcal{O})$	[70]
9	$\Omega_{\underline{0}} \to (\Omega_{\underline{0}})_m^n$	$S_1(\Omega,\Omega)$	$S_1(\mathcal{O},\mathcal{O})$	[70]
10	$S_{fin}(\Omega_{\underline{0}},\Omega_{\underline{0}}^{gp})$	$S_{fin}(\Omega,\Omega^{gp})$	$S_{fin}(\Omega,\Lambda^{gp})$	[48]
11	$G_{fin}(\Omega_{\underline{0}},\Omega_{\underline{0}}^{gp})$	$S_{fin}(\Omega,\Omega^{gp})$	$S_{fin}(\Omega,\Lambda^{gp})$	[48]
12	$\Omega_{\underline{0}} \to \lceil\Omega_{\underline{0}}^{gp}\rceil_m^2$	$S_{fin}(\Omega,\Omega^{gp})$	$S_{fin}(\Omega,\Lambda^{gp})$	[48]
13	$S_1(\Omega_{\underline{0}},\Omega_{\underline{0}}^{gp})$	$S_1(\Omega,\Omega^{gp})$	$S_1(\Omega,\Lambda^{gp})$	[48]
14	$G_1(\Omega_{\underline{0}},\Omega_{\underline{0}}^{gp})$	$S_1(\Omega,\Omega^{gp})$	$S_1(\Omega,\Lambda^{gp})$	[48]
15	$\Omega_{\underline{0}} \to (\Omega_{\underline{0}}^{gp})_m^n$	$S_1(\Omega,\Omega^{gp})$	$S_1(\Omega,\Lambda^{gp})$	[48]

Table 2

The item 9 in Table 2 says that each finite power of a Tychonoff space X has the Hurewicz property if and only if $C_p(X)$ has countable fan tightness as well as the Reznichenko property, while the item 12 states that all finite powers of X have the Gerlits-Nagy property $(*)$ if and only if $C_p(X)$ has countable strong fan tightness and Reznichenko's property. In [67], conditions under which $C_p(X)$ has *only* the Reznichenko property have been found.

An ω-cover $\mathcal{U}$ is *ω-shrinkable* if for each $U \in \mathcal{U}$ there exists a closed set $C(U) \subset U$ such that $\{C(U) : U \in \mathcal{U}\}$ is a closed ω-cover of X.

Theorem 1 ([67]) *For a Tychonoff space X the space $C_p(X)$ has the Reznichenko property if and only if for each ω-shrinkable ω-cover is ω-groupable.*

In [68] it was shown that for each analytic space X the space $C_p(X)$ has the Reznichenko property.

Let us also mention that some other closure properties of $C_p(X)$ spaces can be characterized by covering properties of X (T-tightness and set-tightness [66], selective strictly A-space property in [57], the Pytkeev property in [67]).

Some results regarding the function space $C_k(X)$ are listed in the following table.

	$C_k(X)$	X	Source
1	Countable tightness	k-Lindelöf	[51], [56]
2	SFU	$S_1(\mathcal{K}, \Gamma_k)$	[49]
3	$S_1(\Omega_{\underline{0}}, \Omega_{\underline{0}})$	$S_1(\mathcal{K}, \mathcal{K})$	[40]
4	$S_{fin}(\Omega_{\underline{0}}, \Omega_{\underline{0}})$	$S_{fin}(\mathcal{K}, \mathcal{K})$	[49]

Table 3

We mention also the following result from [40]:

Theorem 2 *If ONE has no winning strategy in the game $G_{fin}(\mathcal{K}, \mathcal{K}^{gp})$ (resp. $G_1(\mathcal{K}, \mathcal{K}^{gp})$) on X, then $C_k(X)$ has the Reznichenko property and countable fan tightness (resp. countable strong fan tightness).*

Similarly to Theorem 1 one proves

Theorem 3 *For a Tychonoff space X the space $C_k(X)$ has the Reznichenko property if and only if for each k-shrinkable k-cover is k-groupable.*

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Clearly, we say that a k-cover $\mathcal{U}$ of a space is k-shrinkable if for each $U \in \mathcal{U}$ there exists a closed set $C(U) \subset U$ such that $\{C(U) : U \in \mathcal{U}\}$ is a closed k-cover of the space.

Quite recently it was shown how some results from Table 2 can be applied to get pure topological characterizations of several classical covering properties in terms of continuous images into the space $\mathbb{R}^\omega$ (Kočinac).

We end this section by some open problems; some of them seem to be difficult.

The next four problems we borrow from [7] (the first two of them were formulated in [37] in a different form).

Problem 4 *Is* $\mathsf{S}_{fin}(\Gamma, \Lambda^{wgp}) = \mathsf{S}_{fin}(\Gamma, \Omega)$?

Problem 5 *If the answer to the previous problem is "not", does* $\mathsf{S}_{fin}(\Gamma, \Lambda^{gp})$ *imply* $\mathsf{S}_{fin}(\Gamma, \Omega)$?

Problem 6 *Is* $\mathsf{S}_1(\Omega, \Lambda^{wgp})$ *stronger than* $\mathsf{S}_1(\Omega, \Lambda)$?

Problem 7 *Is* $\mathsf{S}_1(\Omega, \Omega)$ *stronger than* $\mathsf{S}_1(\Omega, \Lambda^{wgp})$?

A set X of reals is said to be a τ-set (Tsaban) if each ω-cover of X contains a countable family which is a τ-cover?
The next two problems are taken from [82].

Problem 8 *Is the τ-set property equivalent to the γ-set property?*

Problem 9 *Is* $\mathsf{S}_{fin}(\Omega, \mathsf{T})$ *equivalent to the τ-set property?*

According to [20] (resp. [43]) a space X is a k-γ-set (resp. γ_k-set; γ'_k-set) if it satisfies the selection hypotheses $\mathsf{S}_1(\mathcal{K}, \Gamma)$ (resp. each k-cover $\mathcal{U}$ of X contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a γ_k-cover; satisfies $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$). From these two papers we take the next two problems.

Problem 10 *Is the k-γ-set property equivalent to the assertion that each k-cover of X contains a sequence which is a γ-cover?*

Problem 11 *Is the the γ_k-set property equivalent to* $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$?

Problem 12 *Is the converse in Theorem 2 true?*

3   Hyperspaces

In this section we discuss duality between properties of a space X and spaces of closed subsets of X with different topologies illustrating how a selection principle for X can be described by properties of a hyperspace over X. As we shall see, this duality often looks as duality between X and function spaces over X.

By 2^X we denote the family of all closed subsets of a space X. For a subset A of X we put

$$A^c = X \setminus A, \quad A^+ = \{F \in 2^X : F \subset A\}, \quad A^- = \{F \in 2^X : F \cap A \neq \emptyset\}.$$

The most known and popular among topologies on 2^X is the Vietoris topology $\mathsf{V} = \mathsf{V}^- \vee \mathsf{V}^+$, where the *lower Vietoris topology* V^- is generated by all sets A^-, $A \subset X$ open, and the *upper Vietoris topology* V^+ is generated by sets B^+, B open in X.

However, we are interested in other topologies on 2^X.

Let Δ be a subset of 2^X. Then the *upper Δ-topology*, denoted by Δ^+ [59] is the topology whose subbase is the collection

$$\{(D^c)^+ : D \in \Delta\} \cup \{2^X\}.$$

Note: if Δ is closed for finite unions and contains all singletons, then the previous collection is a base for the Δ^+-topology. We consider here two important special cases:

1. Δ is the family of all finite subsets of X, and
2. Δ is the collection of compact subsets of X.

The corresponding Δ^+-topologies will be denoted by Z^+ and F^+, respectively and both have the collections of the above kind as basic sets. The F^+-topology is known as the *upper Fell topology* (or the *co-compact topology*) [27]. The *Fell topology* is $\mathsf{F} = \Delta^+ \vee \mathsf{V}^-$, where V^- is the lower Vietoris topology.

A number of results concerning selection principles in hyperspaces with the Δ^+-topologies obtained in the last years is listed in the following two tables. We would like to say that some results going to a similar direction can be found in [18], [34].

	$(2^X, \mathsf{Z}^+)$	$(\forall Y)(Y$ open in $X)$	Source
1	countable tightness	ω-Lindelöf	folklore
2	FU	$\mathsf{S}_1(\Omega, \Gamma)$	[43]
3	SFU	$\mathsf{S}_1(\Omega, \Gamma)$	[43]
4	countable fan tightness	$\mathsf{S}_{fin}(\Omega, \Omega)$	[21]
5	countable strong fan tightness	$\mathsf{S}_1(\Omega, \Omega)$	[21]
6	$(\forall S \in 2^X)\ \mathsf{S}_{fin}(\Omega_S, \Omega_S^{gp})$	$\mathsf{S}_{fin}(\Omega, \Omega^{gp})$	[42]
7	$(\forall S \in 2^X)\ \mathsf{S}_1(\Omega_S, \Omega_S^{gp})$	$\mathsf{S}_1(\Omega, \Omega^{gp})$	[42]

<div align="center">Table 4</div>

	$(2^X, \mathsf{F}^+)$	$(\forall Y)(Y$ open in $X)$	Source
1	countable tightness	k-Lindelöf	[18]
2	FU	γ_k-set	[43]
3	SFU	$\mathsf{S}_1(\mathcal{K}, \Gamma_k)$	[43]
4	countable fan tightness	$\mathsf{S}_{fin}(\mathcal{K}, \mathcal{K})$	[21]
5	countable strong fan tightness	$\mathsf{S}_1(\mathcal{K}, \mathcal{K})$	[21]
6	$(\forall S \in 2^X)\ \mathsf{S}_{fin}(\Omega_S, \Omega_S^{gp})$	$\mathsf{S}_{fin}(\mathcal{K}, \mathcal{K}^{gp})$	[42]
7	$(\forall S \in 2^X)\ \mathsf{S}_1(\Omega_S, \Omega_S^{gp})$	$\mathsf{S}_1(\mathcal{K}, \mathcal{K}^{gp})$	[42]

<div align="center">Table 5</div>

One more nice property of a space has been considered in a number of recent papers.

Call a space X *selectively Pytkeev* [42] if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ of sets in Ω_x there is an infinite family $\{B_n : n \in \mathbb{N}\}$ of countable infinite sets which is a π-network at x and such that for each n, $B_n \subset A_n$. If all the sets A_n are equal to a set A, one obtains the notion of *Pytkeev spaces* introduced in [60] and then studied in [50] (where the name Pytkeev space was used), [26], [67], [42]. It was shown in [42] that (from a more general result) we have the following.

Theorem 13 *For a space X the following are equivalent:*

(1) $(2^X, \mathsf{F}^+)$ *has the selectively Pytkeev property;*

(2) *For each open set $Y \subset X$ and each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of k-covers of Y there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of infinite, countable sets such that for each n, $\mathcal{V}_n \subset \mathcal{U}_n$ and $\{\cap \mathcal{V}_n : n \in \mathbb{N}\}$ is a (not necessarily open) k-cover of Y.*

<div align="center">12</div>

Similar assertions (from [42]) can be easily formulated for the selectively Pytkeev property in $(2^X, \mathsf{Z}^+)$ and the Pytkeev property in both $(2^X, \mathsf{Z}^+)$ and $(2^X, \mathsf{F}^+)$.

Every (sub)sequential space has the Pytkeev property [60, Lemma 2] and every Pytkeev space has the Reznichenko property [50, Corollary 1.2].

It is natural to ask.

Problem 14 *If $(2^X, \mathsf{F}^+)$ has the Pytkeev property, is $(2^X, \mathsf{F}^+)$ sequential? What about $(2^X, \mathsf{Z}^+)$?*

However, for a locally compact Hausdorff spaces X the countable tightness property, the Reznichenko property and the Pytkeev property coincide in the space $(2^X, \mathsf{F})$ and each of them is equivalent to the fact that X is both hereditarily separable and hereditarily Lindelöf. There are some models of ZFC in which each of these properties is equivalent to sequentiality of $(2^X, \mathsf{F})$ (for locally compact Hausdorff spaces) [42].

At the end of this section we shall discuss the Arhangel'skiĭ α_i properties [1] of hyperspaces according to [22].

A space X has property:

α_1: if for each $x \in X$ and each sequence $(\sigma_n : n \in \mathbb{N})$ of elements of Σ_x there is a $\sigma \in \Sigma_x$ such that for each $n \in \mathbb{N}$ the set $\sigma_n \setminus \sigma$ is finite;

α_2: if for each $x \in X$ and each sequence $(\sigma_n : n \in \mathbb{N})$ of elements of Σ_x there is a $\sigma \in \Sigma_x$ such that for each $n \in \mathbb{N}$ the set $\sigma_n \cap \sigma$ is infinite;

α_3: if for each $x \in X$ and each sequence $(\sigma_n : n \in \mathbb{N})$ of elements of Σ_x there is a $\sigma \in \Sigma_x$ such that for infinitely many $n \in \mathbb{N}$ the set $\sigma_n \cap \sigma$ is infinite;

α_4: if for each $x \in X$ and each sequence $(\sigma_n : n \in \mathbb{N})$ of elements of Σ_x there is a $\sigma \in \Sigma_x$ such that for infinitely many $n \in \mathbb{N}$ the set $\sigma_n \cap \sigma$ is nonempty.

It is understood that
$$\alpha_1 \Rightarrow \alpha_2 \Rightarrow \alpha_3 \Rightarrow \alpha_4.$$

In [22] it is shown a result regarding the Δ^+-topologies one of whose corollaries is the following theorem.

Theorem 15 *For a space X the following statements are equivalent:*

(1) $(2^X, \mathsf{F}^+)$ *is an α_2-space;*

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(2) $(2^X, \mathsf{F}^+)$ *is an α_3-space;*

(3) $(2^X, \mathsf{F}^+)$ *is an α_4-space;*

(4) *For each $S \in 2^X$, $(2^X, \mathsf{F}^+)$ satisfies $\mathsf{S}_1(\Sigma_S, \Sigma_S)$;*

(5) *Each open set $Y \subset X$ satisfies $\mathsf{S}_1(\Gamma_k, \Gamma_k)$.*

At the very end of this section we emphasize the existence of results that have been appeared in the literature [19], [71], [39] in connection with selection principles in the *Pixley-Roy space* $\mathsf{PR}(X)$ over X – the set of finite subsets of X with the topology whose base form the sets

$$[F, U] := \{S \in \mathsf{PR}(X) : F \subset S \subset U\},$$

where F is a finite and U is an open set in X with $F \subset U$.

4 Star and uniform selection principles

We repeat that in this section we assume that all topological spaces are Hausdorff and ω-Lindelöf.

In [38], Kočinac introduced star selection principles in the following way.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X and let $\mathcal{K}$ be a family of subsets of X. Then:

1. The symbol $\mathsf{S}_1^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

> For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{\mathrm{St}(U_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$;

2. The symbol $\mathsf{S}_{fin}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

> For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n \in \mathbb{N}}\{\mathrm{St}(V, \mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$;

3. By $\mathsf{U}_{fin}^*(\mathcal{A}, \mathcal{B})$ we denote the selection hypothesis:

> For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of members of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\mathrm{St}(\cup \mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

4. $\mathsf{SS}_{\mathcal{K}}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(K_n : n \in \mathbb{N})$ of elements of $\mathcal{K}$ such that $\{\mathrm{St}(K_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

When $\mathcal{K}$ is the collection of all one-point (resp., finite) subspaces of X we write $\mathsf{SS}_1^*(\mathcal{A}, \mathcal{B})$ (resp., $\mathsf{SS}_{fin}^*(\mathcal{A}, \mathcal{B})$) instead of $\mathsf{SS}_{\mathcal{K}}^*(\mathcal{A}, \mathcal{B})$.

Here, for a subset A of a space X and a collection $\mathcal{S}$ of subsets of X, $\mathrm{St}(A, \mathcal{S})$ denotes the star of A with respect to $\mathcal{S}$, that is the set $\cup\{S \in \mathcal{S} : A \cap S \neq \emptyset\}$; for $A = \{x\}$, $x \in X$, we write $\mathrm{St}(x, \mathcal{S})$ instead of $St(\{x\}, \mathcal{S})$.

The following terminology we borrow from [38]. A space X is said to have:

1. the *star-Rothberger property* SR,
2. the *star-Menger property* SM,
3. the *strongly star-Rothberger property* SSR,
4. the *strongly star-Menger property* SSM,

if it satisfies the selection hypothesis:

1. $\mathsf{S}_1^*(\mathcal{O}, \mathcal{O})$,
2. $\mathsf{S}_{fin}^*(\mathcal{O}, \mathcal{O})$ (or, equivalently, $\mathsf{U}_{fin}^*(\mathcal{O}, \mathcal{O})$),
3. $\mathsf{SS}_1^*(\mathcal{O}, \mathcal{O})$,
4. $\mathsf{SS}_{fin}^*(\mathcal{O}, \mathcal{O})$,

respectively.

In [15], two star versions of the Hurewicz property were introduced as follows:

SH: A space X satisfies the *star-Hurewicz property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$ $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\mathrm{St}(\cup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n.

SSH: A space X satisfies the *strongly star-Hurewicz property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to $\mathrm{St}(A_n, \mathcal{U}_n)$ for all but finitely many n (i.e. if X satisfies $\mathsf{SS}_{fin}^*(\mathcal{O}, \Gamma)$).

Of course Menger spaces are SSM, amd every SSM space is SM. Similarly for the Hurewicz and Rothberger properties.

There is a strongly star-Menger space which is not Menger, but every metacompact strongly star-Menger space is a Menger space [38]. For paracompact (Hausdorff) spaces the three properties, SM, SSM and M, are equivalent [38]. The same situation is with the classes SSH, SH and H [15].

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The product of two star-Menger (resp. SH) spaces need not be in the same class. But if one factor is compact, then the product is in the same class [38], [15]. A Lindelöf space is not a preserving factor for classes SSM and SSH.

In [38] we posed the following still open problem.

Problem 16 *Characterize spaces X which are SM (SSM, SR, SSR) in all finite powers.*

A partial solution of this problem was given in [15].

Theorem 17 *If each finite power of a space X is SM, then X satisfies $\mathsf{U}^*_{fin}(\mathcal{O}, \Omega)$.*

Theorem 18 *If all finite powers of a space X are strongly star-Menger, then X satisfies $\mathsf{SS}^*_{fin}(\mathcal{O}, \Omega)$.*

In the same paper we read the following two assertions.

Theorem 19 *For a space X the following are equivalent:*

(1) X *satisfies* $\mathsf{U}^*_{fin}(\mathcal{O}, \Omega)$;

(2) X *satisfies* $\mathsf{U}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$.

Theorem 20 *For a space X the following are equivalent:*

(1) X *satisfies* $\mathsf{SS}^*_{fin}(\mathcal{O}, \Omega)$;

(2) X *satisfies* $\mathsf{SS}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$.

So the previous problem can be now translated to

Problem 21 *Does $X \in \mathsf{U}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$ imply that all finite powers of X are star-Menger? Is it true that $\mathsf{S}^*_{fin}(\mathcal{O}, \Omega) = \mathsf{S}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$? Does $X \in \mathsf{SS}^*_{fin}(\mathcal{O}, \mathcal{O}^{wgp})$ imply that each finite power of X is SSM?*

The following result regarding star-Hurewicz spaces

Theorem 22 *For a space X the following are equivalent:*

(1) X *has the strongly star-Hurewicz property;*

(2) X *satisfies the selection principle* $\mathsf{SS}^*_{fin}(\mathcal{O}, \mathcal{O}^{gp})$.

suggests the following

Problem 23 *Is it true that* $\mathsf{S}^*_{fin}(\mathcal{O},\Gamma) = \mathsf{S}^*_{fin}(\mathcal{O},\mathcal{O}^{gp})$*?*

Let us formulate once again a question from [38].

Problem 24 *Characterize hereditarily SM (SSM, SR, SSR, SH, SSH) spaces.*

Let X be a space. Two players, ONE and TWO, play a round per each natural number n. In the n–th round ONE chooses an open cover $\mathcal{U}_n$ of X and TWO responds by choosing a finite set $A_n \subset X$. A play $\mathcal{U}_1, A_1; \cdots ; \mathcal{U}_n, A_n; \cdots$ is won by TWO if $\{\mathrm{St}(A_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a γ-cover of X; otherwise, ONE wins.

Evidently, if ONE has no winning strategy in the strongly star-Hurewicz game, then X is an SSH space.

Conjecture 25 *The strongly star-Hurewicz property of a space X need not imply ONE does not have a winning strategy in the strongly star-Hurewicz game played on X.*

Similar situation might be expected in cases of star versions of the Menger and Rothberger properties and the corresponding games (which can be naturally associated to a selection principle).

But the situation can be quite different in case of zero-dimensional metrizable topological groups (see the next section).

In [41] it was demonstrated that selection principles in uniform spaces are a good application of star selection principles to concrete special classes of spaces. In particular case of topological groups ones obtain nice classes of groups.

Recall that a uniformity on a set X can be defined in terms of uniform covers, and then the uniform space is viewed as the pair $(X, \mathbb{C})$, or in terms of entourages of the diagonal, and then the uniform space is viewed as the pair $(X, \mathbb{U})$ [23]. The first approach is convenient because it allows us to define uniform selection principles in a natural way similar to the definitions of topological selection principles. After that it is easy to pass to $(X, \mathbb{U})$.

Let us explain this on the example of the uniform Menger property. A uniform space $(X, \mathbb{C})$ is *uniformly Menger* or *Menger-bounded* if for each sequence $(\alpha_n : n \in \mathbb{N})$ of uniform covers there is a sequence $(\beta_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\beta_n \subset \alpha_n$ and $\bigcup_{n \in \mathbb{N}} \beta_n$ is a (not necessarily uniform) cover of X.

Theorem 26 *For a uniform space $(X, \mathbb{C})$ the following are equivalent:*

(a) *X has the uniform Menger property;*

(b) *for each sequence $(\alpha_n : n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that $X = \bigcup_{n \in \mathbb{N}} St(A_n, \alpha_n)$;*

(c) *for each sequence $(\alpha_n : n \in \mathbb{N}) \subset \mathbb{C}$ there is a sequence $(\beta_n : n \in \mathbb{N})$ such that for each n β_n is a finite subset of α_n and $X = \bigcup_{n \in \mathbb{N}} St(\cup \beta_n, \alpha_n)$.*

Therefore, we conclude that here we have, in notation we adopted, that $\mathsf{S}_{fin}(\mathbb{C}, \mathcal{O}) = \mathsf{SS}^*_{fin}(\mathbb{C}, \mathcal{O}) = \mathsf{S}^*_{fin}(\mathbb{C}, \mathcal{O})$. In other words, one can say that a uniform space $(X, \mathbb{U})$ is uniformly Menger if and only if for each sequence $(U_n : n \in \mathbb{N})$ of entourages of the diagonal of X there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that $X = \bigcup_{n \in \mathbb{N}} U_n[A_n]$.

It is understood, if a uniform space X has the Menger property with respect to topology generated by the uniformity, then X is uniformly Menger. However, any non-Lindelöf Tychonoff space serves as an example of a space which is uniformly Menger that has no the Menger property. (Similar remarks hold for the uniform Rothberger and uniform Hurewicz properties defined below.) But a regular topological space X has the Menger property if and only if its fine uniformity has the uniform Menger property. Uniform spaces having the uniform Menger property have some properties which are similar to the corresponding properties of totally bounded uniform spaces.

In case of topological groups we have: A topological group $(G, \cdot)$ is *Menger-bounded* if for each sequence $(U_n : n \in \mathbb{N})$ of neighborhoods of the neutral element $e \in G$ there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of G such that $X = \bigcup_{n \in \mathbb{N}} A_n \cdot U_n$. This class of groups was already studied in the literature under the name *o*-bounded groups [32], [33].

More information on Menger-bounded topological groups the reader can find in [32], [33], [8], [11], [12], [83].

Similarly, a uniform space $(X, \mathbb{C})$ is *Rothberger-bounded* if it satisfies one of the three equivalent selection hypotheses: $\mathsf{S}_1(\mathbb{C}, \mathcal{O})$, $\mathsf{SS}^*_1(\mathbb{C}, \mathcal{O})$, $\mathsf{S}^*_1(\mathbb{C}, \mathcal{O})$.

A topological group $(G, \cdot)$ is *Rothberger-bounded* if for each sequence $(U_n : n \in \mathbb{N})$ of neighborhoods of the neutral element $e \in G$ there is a sequence $(x_n : n \in \mathbb{N})$ of elements of G such that $X = \bigcup_{n \in \mathbb{N}} x_n \cdot U_n$.

Finally, a uniform space $(X, \mathbb{C})$ is *uniformly Hurewicz* if for each sequence $(\alpha_n : n \in \mathbb{N})$ of uniform covers of X there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to all but finitely many sets $St(F_n, \alpha_n)$.

It is easy to define *Hurewicz-bounded* topological groups.

The difference between uniform and topological selection principles is big enough [41]. Here we point out some of differences on the example of the Hurewicz properties. (Note that uniformly Hurewicz spaces have many similarities with totally bounded uniform spaces.)

Every subspace of a uniformly Hurewicz uniform space is uniformly Hurewicz. A uniform space X is uniformly Hurewicz if and only if its completion $\tilde{X}$ is uniformly Hurewicz. The product of two uniformly Hurewicz uniform spaces is also uniformly Hurewicz.

Hurewicz-bounded topological groups are preserving factors for the class of Menger-bounded groups [8].

5   Relative selection principles

A systematic study of relative topological properties was started by A.V. Arhangel'skiĭ in 1989 and then continued in a series of his papers and papers of many other authors (see for example [4], [5]).

Let X be a topological space and Y a subspace of X. To each topological property $\mathcal{P}$ (of X) associate a property "relative $\mathcal{P}$" which shows how Y is located in X; thus we speak also that Y *is relatively $\mathcal{P}$ in X*. For $Y = X$ the relative version of a property $\mathcal{P}$ must be just $\mathcal{P}$. In that sense classical topological properties are called *absolute* properties.

A systematic investigation of relative selection principles was initiated by Kočinac (see, [45], [46], [31]). Later on it was shown that relative covering properties described by selection principles, like absolute ones, have nice relations with game theory and Ramsey theory, as well as with with measure-like and basis-like properties in metric spaces and topological groups. We shall see that relative selection principles can be quite different from absolute ones. For example, in [7] it was shown that a very strong relative covering property is not related to a weak absolute covering property. More precisely, it was proved that the Continuum Hypothesis implies the existence of a relative γ-subset X of the real line such that X does not have the (absolute) Menger property $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$. It will be also demonstrated that relative selection principles strongly depend on the nature of the basic space.

Notice that much still needs to be investigated regarding the relative selection principles in connection with "non-classical" selection principles.

Let X be a space and Y a subset of X. We use the symbol $\mathcal{O}_X$ to denote the family of open covers of X and the symbol $\mathcal{O}_Y$ for the set of covers of Y by sets open in X. Similar notation will be used for other families of covers.

In this notation we have:

- *Y is relatively Menger in X* [45]

- *Y is relatively Rothberger in X* [45]

- *Y is relatively Hurewicz in X* [31], [7]

- *Y is a relative γ-set in X* [46]

if the following selection principle is satisfied

- $\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_Y)$

- $\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_Y)$

- $\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_Y^{gp})$

- $\mathsf{S}_1(\Omega_X, \Gamma_Y)$.

When $Y = X$ we obtain considered absolute versions of selection principles.

In Section 2 we saw that there is a nice duality between covering properties of a Tychonoff space X and closure properties of function spaces $\mathsf{C}_p(X)$ and $\mathsf{C}_k(X)$. In what follows we show that similar duality exists between relative selection principles and closure properties of mappings. For a Tychonoff space X and its subspace Y the restriction mapping $\pi : \mathsf{C}_p(X) \to \mathsf{C}_p(Y)$ is defined by $\pi(f) = f \restriction Y$, $f \in \mathsf{C}_p(X)$.

Relative Menger property

If $f : X \to Y$ is a continuous mapping, then we say that f has *countable fan tightness* if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ of elements of Ω_x there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subset A_n$ and $\bigcup_{n \in \mathbb{N}} f(B_n) \in \Omega_{f(x)}$.

The following theorem from [45] gives a relation between relative Menger-like properties and fan tightness of mappings.

Theorem 27 *For a Tychonoff space X and a subspace Y of X the following are equivalent:*

(1) *For all $n \in \mathbb{N}$, Y^n is Menger in X^n;*

(2) $\mathsf{S}_{fin}(\Omega_X, \Omega_Y)$ *holds;*

(3) *The mapping π has countable fan tightness.*

In [6] the relative Menger property was further considered and the following theorem proved (compare with item 11 in Table 1):

Theorem 28 *Let X be a Lindelöf space. Then for each subspace Y of X the following are equivalent:*

(1) $\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_Y)$;

(2) *ONE has no winning strategy in $\mathsf{G}_{fin}(\mathcal{O}_X, \mathcal{O}_Y)$;*

(3) *For each natural number m, $\Omega_X \to \lceil \mathcal{O}_Y \rceil_m^2$.*

The following result from [10] is of the same sort and is a relative version of a result from [7].

Theorem 29 *Let X be a space with the the Menger property $\mathsf{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ and Y a subspace of X. The following are equivalent:*

(1) $\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_Y^{wgp})$;

(2) *ONE has no winning strategy in the game $\mathsf{G}_{fin}(\Omega_X, \mathcal{O}_Y^{wgp})$;*

(3) *For each $m \in \mathbb{N}$, $\Omega_X \to \lceil \mathcal{O}_Y^{wgp} \rceil_m^2$.*

The relative Menger property in metric spaces has basis-like and measure-like characterizations as it was shown in [9] and [10]. Relative Menger-like properties in topological groups also have very nice characterizations [8].

To formulate results in this connection we need some terminology.

In [52] Menger introduced a property for metric spaces (X, d) that we call the *Menger basis property*: For each base $\mathcal{B}$ in X there is a sequence $(B_n : n \in \mathbb{N})$ in $\mathcal{B}$ such that $\lim_{n\to\infty} diam(B_n) = 0$ and the set $\{B_n : n \in \mathbb{N}\}$ is an open cover of X. As we mentioned in Introduction, in [35] W. Hurewicz proved that a metrizable space X has the Menger basis property with respect to all metrics on X generating the topology of X if and only if it has the Menger property $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$.

Say that a subspace Y of a metric space (X, d) has the *Menger basis property in X* if for each base $\mathcal{B}$ in X there is a sequence $(B_n : n \in \mathbb{N})$ in $\mathcal{B}$ such that $\lim_{n\to\infty} diam(B_n) = 0$ and the set $\{B_n : n \in \mathbb{N}\}$ is an open cover of Y.

The following definition is motivated by the definition of strong measure zero sets introduced by Borel in [16] (see the subsection on relative Rothberger property).

A metric space (X, d) has *Menger measure zero* if for each sequence $(\epsilon_n : n \in \mathbb{N})$ of positive real numbers there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that:

(i) for each n, $\mathcal{V}_n$ is a finite family of subsets of X;

(ii) for each n and each $V \in \mathcal{V}_n$, $\text{diam}_d(V) < \epsilon_n$;

(iii) $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is an open cover of X.

Combining some results from [9] and [10] we have the following theorem.

Theorem 30 *Let (X, d) be a separable zero-dimensional metric space and let Y be a subspace of X. The following statements are equivalent:*

(1) *Y is relatively Menger in X;*

(2) *Y has the Menger basis property in X;*

(3) *Y has Menger measure zero with respect to each metric on X which gives X the same topology as d.*

Let $(G, \cdot)$ be a topological group and H its subgroup. Denote by $\mathsf{M}(G, H)$ the following game for two players, ONE and TWO, which play a round for each $n \in \mathbb{N}$. In the n-th round ONE chooses a neighborhood U_n of the neutral element of G and then TWO chooses a finite set $F_n \subset G$. Two wins a play $U_1, F_1; U_2, F_2; \ldots$ if and only if $\{F_n \cdot U_n : n \in \mathbb{N}\}$ covers H. (It is a relative version of a game first mentioned in [33].)

In [8], the following result regarding Menger-like properties for topological groups has been obtained.

Theorem 31 *Let G be a zero-dimensional metrizable group and let H be a subgroup of G. The following assertions are equivalent:*

(1) *H is Menger-bounded;*

(2) *H is Menger-bounded in G;*

(3) *ONE has no winning strategy in the game $\mathsf{M}(H, H)$;*

(4) *H has the relative Menger property in G;*

(5) *H has Menger measure zero with respect to all metrizations of G.*

Relative Hurewicz property

Recall that a subspace Y of a space X is relatively Hurewicz in X if the selection principle $\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_Y^{gp})$ holds.

Following [31] and [7] we say that a continuous mapping $f : X \to Y$ has the *selectively Reznichenko property* if for each sequence $(A_n : n \in \mathbb{N})$ from Ω_x there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n, B_n is a finite subset of A_n and $\bigcup_{n \in \mathbb{N}} B_n \in \Omega_{f(x)}^{gp}$.

The theorem below is a combination of results from [31] and [48] and gives a characterization of the relative Hurewicz property in all finite powers [31].

Theorem 32 *For a Tychonoff space X and its subspace Y the following are equivalent:*

(1) *π has the selectively Reznichenko property;*

(2) *For each $n \in \mathbb{N}$, Y^n has the Hurewicz property in X^n;*

(3) *ONE has no winning strategy in $\mathsf{G}_{fin}(\Omega_X, \Omega_Y^{gp})$;*

(4) *For each $m \in \mathbb{N}$, $\Omega_X \to \lceil \Omega_Y^{gp} \rceil_m^2$.*

The relative Hurewicz property has also a game-theoretic and Ramsey-theoretic description [7].

Theorem 33 *Let X be a Lindelöf space. Then for each subspace Y of X the following are equivalent:*

(1) *$\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_Y^{gp})$;*

(2) *ONE has no winning strategy in $\mathsf{G}_{fin}(\Omega_X, \mathcal{O}_Y^{gp})$;*

(3) *For each $m \in \mathbb{N}$, $\Omega_X \to \lceil \mathcal{O}_Y^{gp} \rceil_m^2$.*

Following [7], we are going now to show that the relative Hurewicz property for metric spaces can be characterized by basis-like and measure-like properties.

Let (X, d) be a metric space and Y a subspace of X. Then:

Y has the *Hurewicz basis property* in X if for any basis $\mathcal{B}$ of X there is a sequence $(U_n : n \in \mathbb{N})$ in $\mathcal{B}$ such that $\{U_n : n \in \mathbb{N}\}$ is a groupable cover of Y and $\lim_{n \to \infty} diam_d(U_n) = 0$.

Y has *Hurewicz measure zero* (in X) if for each sequence $(\epsilon_n : n \in \mathbb{N})$ of positive real numbers there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that:

(i) for each n, $\mathcal{V}_n$ is a finite family of subsets of X;

(ii) for each n and each $V \in \mathcal{V}_n$, $\mathrm{diam}_d(V) < \epsilon_n$;

(iii) $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a groupable cover of X.

Theorem 34 ([7]) *Let (X, d) be a metric space and let Y be a subspace of X. The following statements are equivalent:*

(1) *Y is relatively Hurewicz in X;*

(2) *Y has the Hurewicz basis property in X.*

If (X, d) is zero-dimensional and separable, then conditions (1) and (2) are equivalent to

(3) *Y has Hurewicz measure zero with respect to each metric on X which gives X the same topology as d does.*

For special topological groups we have interesting characterizations of relative versions of Hurewisz-like properties [8]. The following result shows again how relative properties depend on the structure of the basic space.

Theorem 35 *For a subgroup $(G, +)$ of $(^\omega\mathbb{Z}, +)$ the following are equivalent:*

(1) *G is Hurewicz-bounded;*

(2) *G has Hurewicz measure zero in the Baire metric on $^\omega\mathbb{Z}$;*

(3) *G has the relative Hurewicz property in $^\omega\mathbb{Z}$.*

Similar results for the selection principle $\mathsf{S}_{fin}(\Omega_X, \mathcal{O}_Y^{wgp})$ can be found in [6], [9] and [10].

Relative Rothberger property

A continuous mapping $f : X \to Y$ is said to have *countable strong fan tightness* if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ from Ω_x there exist $x_n \in A_n$, $n \in \mathbb{N}$, such that $\{f(x_n) : n \in \mathbb{N}\} \in \Omega_{f(x)}$.

Here is a theorem from [45] which gives a characterization of the relative Rothberger property.

Theorem 36 *If Y is a subset of a Tychonoff space X then the following are equivalent:*

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(1) *For each $n \in \mathbb{N}$, Y^n has the Rothberger property in X^n;*

(b) *The selection principle $\mathsf{S}_1(\Omega_X, \Omega_Y)$ holds;*

(c) *The mapping $\pi : \mathsf{C}_p(X) \to \mathsf{C}_p(Y)$ has countable strong fan tightness.*

In [16] Borel defined a notion for metric spaces (X, d) nowadays called *strong measure zero*. $Y \subset X$ is *strong measure zero* if for each sequence $(\epsilon_n : n \in \mathbb{N})$ of positive real numbers there is a sequence $(U_n : n \in \mathbb{N})$ of subsets of X such that for each n, U_n has diameter $< \epsilon_n$ and $\{U_n : n \in \mathbb{N}\}$ covers Y.

In [54] it was shown that a metric space X has the (absolute) Rothberger property if and only if it has strong measure zero with respect to each metric on X which generates the topology of X.

A result from [72] states that if Y is a subset of a σ-compact metrizable space X, then Y has the relative Rothberger property in X if and only if Y has strong measure zero with respect to each metric on X which generates the topology of X. It is also shown that in this case the previous two conditions are equivalent to each of the next two conditions:

- ONE has no winning strategy in the game $\mathsf{G}_1(\Omega_X, \mathcal{O}_Y)$;

- For each $m \in \mathbb{N}$, $\Omega_X \to (\mathcal{O}_Y)^2_m$.

To state a result similar to (a part of) Theorem 30 we need the following notion.

A subset Y of a metric space (X, d) has the *Rothberger basis property* in X if for each base $\mathcal{B}$ in X and for each sequence $(\epsilon_n : n \in \mathbb{N})$ of positive real numbers there is a sequence $(B_n : n \in \mathbb{N})$ of elements of $\mathcal{B}$ such that $diam_d(B_n) < \epsilon_n$, and $\{B_n : n \in \mathbb{N}\}$ covers Y.

Theorem 37 *Let X be a metrizable space, Y a subspace of X. Then the following are equivalent:*

(1) *Y has the relative Rothberger property in X;*

(2) *Y has the Rothberger basis property in X with respect to all metrics generating the topology of X.*

For Rothberger-bounded subgroups of the set of real numbers we have the following description [8]:

Theorem 38 *For a subgroup $(G, +)$ of $(\mathbb{R}, +)$ the following are equivalent:*

(1) G *is Rothberger-bounded;*

(2) G *has strong measure zero in* $\mathbb{R}$*;*

(3) G *has the relative Rothberger property in* $\mathbb{R}$*.*

Relative Gerlits-Nagy property $(*)$

It is the property $\mathsf{S}_1(\Omega_X, \mathcal{O}^{gp})$, where Y is a subspace of a space X.

We state here only one statement regarding this property in metric spaces.

A subspace Y of a metric space (X, d) has the *Gerlits-Nagy basis property* in X if for each base $\mathcal{B}$ for the topology of X and for each sequence $(\epsilon_n : n \in \mathbb{N})$ of positive real numbers there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n, $B_n \in \mathcal{B}$ and $diam(B_n) < \epsilon_n$, and $\{B_n : n \in \mathbb{N}\}$ is a groupable cover of Y.

The following result is from [9] and [10].

Theorem 39 *Let X be an infinite σ-compact metrizable space and let Y be a subspace of X. The following statements are equivalent:*

(1) $\mathsf{S}_1(\Omega_X, \mathcal{O}_Y^{gp})$*;*

(1) *ONE has no winning strategy in the game* $\mathsf{G}_1(\Omega_X, \mathcal{O}_Y^{gp})$*;*

(1) *For each positive integer m, $\Omega_X \to (\mathcal{O}_Y^{gp})_m^2$;*

(1) *Y has the Gerlits-Nagy basis property in X with respect to all metrics generating the topology of X.*

Let us point out that similar assertion (for σ-compact metrizable spaces) is true for the principle $\mathsf{S}_1(\Omega_X, \mathcal{O}_Y^{wgp})$.

Relative γ-sets

A subspace Y of a space X is a relative γ-set in X if the selection hypothesis $\mathsf{S}_1(\Omega_X, \Gamma_Y)$ holds [46].

Clearly, every γ-set is also a relative γ-set, but the converse is not true. The relative γ-set property is hereditary, while the Gerlits-Nagy γ-set property is not hereditary [46]. Relative γ-sets of real numbers have strong measure zero. By the well known facts on strong measure zero sets (Borel's conjecture that no uncountable set of real numbers has strong measure zero

is undecidable in ZFC), the question if there is an uncountable relative γ-set of real numbers is undecidable in ZFC.

The relative γ-set property, as other relative covering properties defined in terms of selection principles, depend on the basic space X. Recently, A.W. Miller [53] considered relative γ-sets in 2^ω and ω^ω. He defined two cardinals $\mathfrak{p}(2^\omega)$ (resp. $\mathfrak{p}(\omega^\omega)$) to be the smallest cardinality of the set X in 2^ω (resp. in ω^ω) which is not a relative γ-set in the corresponding space, observed that $\mathfrak{p} \le \mathfrak{p}(\omega^\omega) \le \mathfrak{p}(2^\omega)$ and proved that it is relatively consistent with ZFC that $\mathfrak{p} = \mathfrak{p}(\omega^\omega) < \mathfrak{p}(2^\omega)$ and $\mathfrak{p} < \mathfrak{p}(\omega^\omega) = \mathfrak{p}(2^\omega)$.

Here $\mathfrak{p}$ is the pseudointersection number and is the cardinality of the smallest non-gamma set (according to a result from [30]; see also [37]).

To characterize relative γ-sets in $\mathbb{R}$ we need the following notion [46].

A continuous mapping $f : X \to Y$ is said to be *strongly Fréchet* if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ in Ω_x there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n, B_n is a finite subset of A_n and the sequence $(f(B_n) : n \in \mathbb{N})$ converges to $f(x)$.

Theorem 40 ([46]) *For a Tychonoff space X and its subspace Y the following are equivalent:*

(1) *Y is a γ-set in X;*

(2) *For each $n \in \mathbb{N}$, Y^n is a γ-set in X^n;*

(3) *The mapping $\pi : \mathsf{C}_p(X) \to \mathsf{C}_p(Y)$ is strongly Fréchet.*

Relative SSH spaces

We close this section by a relative version of a star selection principle from Section 4 and considered in [15]. Once more we conclude that relative selection principles are very different from absolute ones.

Let Y be a subspace of a space X. We say that Y *is strongly star-Hurewicz in X* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that each point $y \in Y$ belongs to all but finitely many sets $\mathrm{St}(A_n, \mathcal{U}_n)$.

There is a strongly star-Menger space X and a subspace Y of X such that Y is relatively strongly star-Hurewicz in X but not (absolutely) strongly star-Hurewicz. The space X is the Mrówka-Isbel space $\Psi(\mathcal{A})$ [23], Y is the subspace $\mathcal{A}$, where $\mathcal{A}$ is an almost disjoint family of infinite subsets of positive integers having cardinality $< \mathfrak{b}$ (see [15]).

Notice that the following two relative versions of the SSH property could be investigated.

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(1) For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of Y by sets open in X there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that each point $y \in Y$ belongs to all but finitely many sets $\mathrm{St}(A_n, \mathcal{U}_n)$.

(2) For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of Y such that each point $y \in Y$ belongs to all but finitely many sets $\mathrm{St}(A_n, \mathcal{U}_n)$.

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